On the Use of Financial Data as a Random Beacon

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Abstract

In standard voting procedures, random audits are one method for increasing election integrity. In the case of cryptographic (or end-to-end) election verification, random challenges are often used to establish that the tally was computed correctly. In both cases, a source of randomness is required. In two recent binding cryptographic elections, this randomness was drawn from stock market data. This approach allows anyone with access to financial data to verify the challenges were generated correctly and, assuming market fluctuations are unpredictable to some degree, the challenges were generated at the correct time. However the degree to which these fluctuations are unpredictable is not known to be sufficient for generating a fair and unpredictable challenge. In this paper, we use tools from computational finance to provide an estimate of the amount of entropy in the closing price of a stock. We estimate that for each of the 30 stocks in the Dow Jones industrial average, the entropy is between 6 and 9 bits per trading day. We then propose a straight-forward protocol for regularly publishing verifiable 128-bit random seeds with entropy harvested over time from stock prices. These "beacons" can be used as challenges directly, or as a seed to a deterministic pseudorandom generator for creating larger challenges.

1 Introductory Remarks

In many countries, including the United States, electronic elections have become predominant. Some electronic voting technologies additionally provide a paper-based record of each vote—*e.g.,* optical scan and DRE machines with voter-verified paper audit trails. This paper record can be compared with the electronic tally, and while both tallies could be consistently manipulated, this check does provide some level of assurance that the electronic tally is correct. It is not feasible to perform this check for all precincts; however by randomly selecting a small number of precincts in an unpredictable way, excellent statistical assurance of the consistency of these tallies can be achieved.

This mathematical approach to protecting against errors and manipulation in elections can be taken a step further. Cryptographic voting systems use mathematical techniques throughout the entire voting process to provide a stronger notion of integrity: even if all the records of all the ballots are subverted, the manipulation will be reliably detected. In many cryptographic systems, assurance is established by challenging the system to prove computations were performed correctly. As with choosing precincts, challenges in cryptographic voting must be random.

This paper is concerned with choosing a source of randomness to generate these random selections or challenges. We will speak abstractly and phrase the nature of the random action as selecting *units* to *audit* (following [36]). It is hopefully intuitive that if the source of randomness used to select units is predictable or can be influenced, the statistical assurance loses strength.

One proposal for an unpredictable source of randomness is financial data (we review a number of other proposed methods in the next section). While not broadly adopted, this technique has been used in at least two binding cryptographic elections: a university election with Punchscan [21] and a municipal election with Scantegrity II [11]. In both cases, cryptographic tools were used to transform the random financial data into a useful form. This may lead some to object to using this type of protocol in non-cryptographic voting contexts, like precinct selection. Our position here is neutral—its application to cryptographic voting is motivation enough for this study, and we note that if cryptographic techniques were permissible for use, then this approach does provide a more publicly verifiable challenge than, say, rolling dice in a room.

It may be acceptable to the reader that financial data, like stock market prices, exhibits some unpredictable behaviour, but it is not likely intuitive how much randomness there is. In this paper, we combine a model from computational finance with techniques from information theory to estimate the amount of entropy in the daily closing prices of a number of common stocks (the same set used in the Scantegrity II election). In our sample, a typical closing price has an estimated 6–9 bits of entropy. When we consider the joint entropy between two highly correlated stocks, we find that the joint information is less than a single bit, suggesting that using a portfolio of stocks is a good method of increasing the pool of entropy.

Finally, we present a straight-forward protocol that can be used to take the prices in their raw form and produce a near-uniform random bit string. Our intention is that some entity will publish the output from our protocol on each day that the market closes as a service. As an incentive to offering this service, we allow the entity to mix in their own randomness. This is done in a way that is transparent and verifiable to everyone to ensure that the entity will be caught if it misbehaves. The output can then be used either directly to select units for auditing, or if many bits are required, the output can be used to seed to a pseudorandom generator. The output is also useful for non-voting cryptographic protocols where a beacon or common reference string is required.

Contributions. In this paper, we present the following:

- a general technique for estimating the amount of entropy in a stock market closing price that is extensible to any simulatable model in computational finance,
- a set of empirical estimates of the entropy in a representative portfolio of stocks, under the conservative assumption that the Black-Scholes model holds for market behaviour,
- a demonstration of the impact on entropy that the four parameters—drift, diffusion, initial price, and time period—have within the range of our data,
- an estimate of the mutual information shared by correlated stocks within the same business sector, and
- a protocol for regularly publishing verifiable random bitstrings based on stock prices.

2 Related Work

2.1 Public Randomness in Cryptography

Many cryptographic protocols require randomness. For most applications, the randomness is only for private use. However, in some scenarios, publicly verifiable randomness can be useful. Two examples are fair exchange and interactive proofs.

In multiparty protocols, it is sometimes advantageous for a malicious party to abort early after they have received information from the other parties and prior to sending their own. An early proposal to mitigate this threat in the case of contract signing, due to Rabin, introduced the notion of a *beacon*, which is an unpredictable and random value published at regular intervals by a trusted third party [35]. Later, contract signing protocols without beacons were proposed [22].

In interactive proofs, including those with the zero knowledge property, a verifier interacts with a prover to become convinced of some fact. A common form of an interactive proof follows three steps (sometimes called a sigma protocol): the prover commits to some information, the verifier generates a random challenge, and the prover, bound to the information they committed to, must respond convincingly to the challenge. In the earliest model, due to Babai, the verifier was bound to only producing random bits visible to both the prover and verifier— *public coins* [5]. Goldwasser and Sipser later proved that this class is, for all practical purposes, equivalent to a class with coins that are visible to only the verifier: *private coins* [26]. A party independent of the verifier can also become convinced of the proof if they believe the coins were random and unpredictable to the prover. In protocols where the verifier's role is only to choose a random challenge, Fiat and Shamir showed that the verifier can sometimes be eliminated entirely by hashing a transcript of the protocol up to that point and using the output as the challenge, making the proof non-interactive [24]. More recently, Groth and Sahai (among others) demonstrated non-interactive proof systems without random challenges at all, based on bilinear pairings [27].

When random (but not necessarily unpredictable) bits are assumed to be available for reference, a protocol can be described in the *common reference string* (CRS) model [8]. In addition to making interactive proofs non-interactive, CRS can be used to achieve security under composition. While the CRS model is very rigid about the distribution the CRS is drawn from, recent work due to Canetti *et al.* has shown results with an imperfect CRS called a *sunspot* [10].

2.2 Public Randomness in Elections

Public randomness is used in post-election auditing procedures, usually for the selection of precincts for manual recounts. Cordero *et al.* establish a set of desirable properties for generating the randomness: it should be simple to understand, efficient to execute, difficult to manipulate, and verifiable after the fact [16]. The authors consider a number of possible mechanisms: cryptographic coin tossing protocols, drawing tickets from a hat, lottery machines, random number charts, shuffling cards, flipping physical coins, and rolling dice. They then devise an algorithm for random precinct selection using dice.

Three main criticisms of the dice-based approach are available. Clark *et al.* point out that protocol is only verifiable to those in the room [12], Calandrino *et al.* point out that the number of rolls can become infeasible for reasonably-sized districts [13], and both note the difficulty of determining whether the dice are fair.[1] Calandrino *et al.* further suggest a hybrid protocol that involves both dice and random draws in order to generate a seed, which is expanded with a cryptographically secure pseudorandom number generator (CS-PRNG). They allow the procedure to be video recorded to expand verification to those not present.

Of the other mechanisms suggested by Cordero *et al.*, at least two others have been examined further. Hall finds that a particular real-world use of a lottery machine, where numbered ping pong balls were drawn from a hopper, yields non-uniform results and proposes a fix [29]. Rescorla examines the use of random number charts, where a seed is used to index into a random spot in the chart [36]. Since the chart is limited in size, the seed is low-entropy (*e.g.,* 16 bits) and the state space is small. This has various consequences, including difficulty in the selection of a pseudorandom stream that is statistically independent of other possible streams. Rescorla finds that using the same seed with a PRNG yields better properties. Clark *et al.* use a different statistical approach but also find that low-entropy seeds expanded with a PRNG can be secure.

2.3 Cryptographic Elections

The use of algorithmic or cryptographic techniques, like (CS-)PRNGs, has been criticized for potential use in normal elections as being difficult to understand and computer-reliant [30]. However in cryptographic elections, where extensive cryptographic techniques are already being used, it is obviously congruent. In cryptographic voting systems, interactive proofs and arguments are typically used. For this reason, if one accepts the random oracle assumption, the mentioned Fiat-Shamir heuristic is often the easiest mechanism to implement. Systems like Helios, used in a binding student election [2], use this approach [1]. However Fiat-Shamir is suitable only when the challenges are drawn from a large space—a space larger than that which can be exhaustively searched [25]. In both Punchscan and Scantegrity, the numbers are used to select half of 10 to 20 units (committed to by the election officials) to audit in a post-election, cut-and-choose argument that proves the tally was not manipulated by the officials [21, 11].[2] For this particular audit, Fiat-Shamir is not useable, motivating the use of public randomness instead.

2.4 Stock Market for Public Randomness

Stock market data has been suggested for use as public randomness in several publications. Waters *et al.* propose a service called a *bastion*, which is similar to Rabin's notion of a beacon, only it produces random cryptographic puzzles instead of random numbers [39]. These puzzles are issued by online services to clients to solve prior to gaining access, which helps prevent denial of service attacks. The authors relate bastions to beacons and specifically suggest the option of financial market data for implementing a beacon. A subset of these authors, Halderman *et al.*, later propose and implement a more general framework, Combine, for harvesting challenges from various online data sources, that can include financial data, to thwart Sybil attacks [28].

Stock market data has also been suggested for randomly selecting an IETF nominating committee, along with lottery numbers and sporting outcomes [19]. However in a later update, financial data was specifically advised to be discontinued because it is not always reported consistently from all sources [20]. We address this issue later in Section 3.4.

Finally, stock market data was proposed by Clark *et al.* for use in the Punchscan cryptographic voting system [12]. This approach of using stock market data is preserved in the Scantegrity II system, which is related to Punchscan through contributors and code-base. However for Scantegrity II, Rivest implements a novel protocol for converting the portfolio of closing prices into pseudorandom bits.[3] Two binding elections were conducted using stock market data for public randomness: a student election in Canada in 2007 with Punchscan and a municipal election in Maryland in 2009 with Scantegrity II.

3 Model and Assumptions

Our primary interest in this paper is the use of financial data as a source of entropy for creating random and unpredictable challenges. If they are truly unpredictable, these challenges can be used in cryptographic voting protocols, particularly zero-knowledge and cut-and-choose

[1] Interestingly, this problem has a solution although it appears to never be mentioned in the voting literature. The solution builds on von Neumann's famous result for generating a fair coin from an unfair coin: flip the unfair coin twice, if it is both heads (HH) or both tails (TT), discard the trial (output $\perp$). If it is heads followed by tails (HT), output a heads (H), and if it is TH, output T. This can generalized to n-sided dice [31].

[2] Here, the cut-and-choose argument follows the same three-step procedure of a zero-knowledge sigma protocol: commit stage, challenge stage, and response change. It is not a zero-knowledge proof for technical reasons that are beyond the scope of this paper.

[3] R. Rivest, 2009. See: `get_latest_djia_stock_prices.py`, `pre_election_audit.py`, and `post_election_audit.py`. `https://scantegrity.org/svn/data/takoma-nov3-2009/PUBLIC/PUBLIC/`.

protocols, to eliminate the need of a verifier to generate the challenges. While this approach could be used anywhere a challenge is needed, it is especially relevant when the efficient Fiat-Shamir heuristic cannot be used. We are motivated to examine stock prices because of their actual use in binding elections.

Note that random and unpredictable mean subtly different things. If an adversary is able to *set* a challenge to a known value, it is not unpredictable to her. However the value may be statistically random and have high entropy in that sense. Our approach is careful to model the uncertainty of the adversary (or anyone) in predicting the outcome of a stock price. This is different from directly computing the statistical randomness contained in financial data, which could lead to a wrong estimate of the *adversarial* uncertainty.

In this section, we introduce the model that we will use to simulate the movement of stock prices. We assume the reader has no background in computational finance. This model will be used to estimate of the amount of uncertainty in a stock price in the next section. Here, we also address some other potential concerns with using stock prices; namely, manipulation and consistent reporting.

3.1 Terminology

Let S_{t_i} be the price of a stock at some time t_i. For time-period T, let S_0 and S_T represent the stock's initial and final price, respectively. Define r to be the risk-free interest rate: the interest-rate on a safe asset, like a government bond, with value β_{t_i}. If r is continuously compounded over period T, a bond initially worth β_0 becomes worth $\beta_T = \beta_0 e^{rT}$.

A *Wiener process*, W_t, is a continuous time process with the following properties:

- $W_0 = 0$.
- $W_t \sim N(0,t)$, where $N(0,t)$ is a normal distribution with mean 0 and variance t.
- For all intervals in time, $t_b - t_a$, $\Delta W = W_b - W_a$ is independent from all other (non-overlapping) intervals.

A Wiener process is also known as standard Brownian motion, and since future values of the process depend only on the current value, it is an example of a Markov process. *Geometric Brownian Motion (GBM)* for random variable X_t adds a linear constant, μ, to the process, which is known as drift, and also scales the variance of the Wiener process, W_t by the constant σ, known as diffusion:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

When X_t represents the value of some instrument, μ is termed the growth rate or expected rate of return. When this is greater than the risk-free rate, it is termed an excess return. The diffusion, σ, is termed the volatility. When volatility is estimated based on past performance of the stock, it is termed historic volatility.

3.2 Black-Scholes Model

The *Black-Scholes model*, or Black-Merton-Scholes, is used to model financial markets and determine the value of derivatives (a financial instrument whose value is dependent on the value of an underlying asset) [7, 32]. We only use the model to study the movement of the underlying asset; in this case, the assets are stocks. The model is based on the following assumptions.

1. There are no transaction costs or dividends.
2. Over time, the asset price is a real number: $S_t \in \mathbb{R}$.
3. Over time, the asset price follows a GBM: $\mathrm{d}S_t = \mu\, S_t\, \mathrm{d}\, t \;+\; \sigma\, S_t\, \mathrm{d}\, W_t$.
4. Over time, μ and σ are constant valued.
5. There are no arbitrage opportunities.[4]

Nearly every mathematical model of a financial market has its criticisms. Black-Scholes is very well-known[5] but has also been controversial, primarily for being too tame of a representation for markets like stocks or commodities. For pricing derivatives, underestimating the volatility of the market can lead to catastrophic loss and thus using models with higher volatility, like Jump-Diffusion models, are a more conservative approach (however they also lead to less competitive pricing) [38]. In our case, we are using stock prices to generate random challenges. It should be intuitive that the entropy in a stock price will increase with higher market volatility (if not, we show this in section 4.4), and so the conservative approach for our purposes is the exact opposite. We use Black-Scholes model because, if anything, it errs on the side of not having enough volatility and therefore will be useful in determining a plausible lower-bound on entropy.[6]

3.3 Market Manipulation

Since trades are the mechanism that moves the price of a stock in the real-world, the closing price of a stock

[4] Through an argument omitted here (see [38]), this effectively means μ is modelled with r (the risk-free rate) when pricing options. Since we are not pricing options, we use historic volatility to estimate values of μ.

[5] Merton and Scholes received a Nobel prize for developing it. Black unfortunately did not live long enough to join them.

[6] If, alternatively, the stock market is more predictable than Black-Scholes, our estimates will be wrong. We do not consider this at length as it is not an advocated position, even by a minority, within the financial community, nor is it supported by the empirical evidence.

could be manipulated through unnatural sales or purchases, particularly near the closing time of the market. This manipulation is theoretically possible and has been performed on exchanges in developing markets; however there is broad agreement that it is difficult to perform on stocks one might find on an established exchange like the NYSE or NSDAQ. We also note that it is illegal in all major exchange markets.

If manipulation were possible, it could be used in the context of cryptographic voting for the following attack: prior to committing to the election data, an adversary creates a guess for the closing price of each stock that will be used. The adversary then determines what the challenge would be if these guesses turn out to be correct, and hides any electoral fraud in the units that will not audited under this envisioned challenge. The manipulated data is then committed to. Later, the adversary buys shares to raise the price of any stock that is below the guessed value and sells shares (short sells, if the adversary does not hold the shares) to lower the price of stocks over its targeted price. If all the prices close exactly on target, the fraud will escape detection.

There are a large number of practical issues with such an attack (its detectability and illegality, the high volatility of prices near closing time, the use of matching algorithms in determining a closing price, regulation concerning short selling after downticks, etc.) but it is theoretically possible. Even if volume is factored into the randomness, the adversary could choose an unusually high target volume to avoid overshooting it while manipulating.

Market manipulation is considered for different reasons by the financial community. One type of financial derivative that can be purchased is a *barrier option*, which operates like a regular stock option (vanilla option) with the added condition that if the underlying asset goes above (or below) a predetermined price (the barrier) at some time interval (such as the daily closing price), the option becomes worthless. If market manipulations were feasible, they could be used to bump stocks over (or under) a barrier.

There is broad agreement that this type of manipulation is difficult if the market is volatile and liquid, and/or if the barrier event must happen multiple times (so-called Parisian options [15]). An empiric study of manipulations in the NYSE, NSDAQ, and other markets during the period of 1990–2001 confirms that manipulations of this type are rare and confined to illiquid stocks [3]. Despite the theoretic possibility of manipulation, barrier options continue to be written/held by banks and investors [37]. We also note that these manipulations have a relatively crude goal: to move the stock up or down. In the case before us, manipulations would have be highly calibrated to result in a stock landing on an exact price (to the nearest cent). For the reasons outlined in this section, we consider manipulation infeasible with the selection of liquid stocks on an established exchange. The election we are studying used the 30 stocks in the Dow Jones industrial average, which meets our criteria.

3.4 Official Closing Price

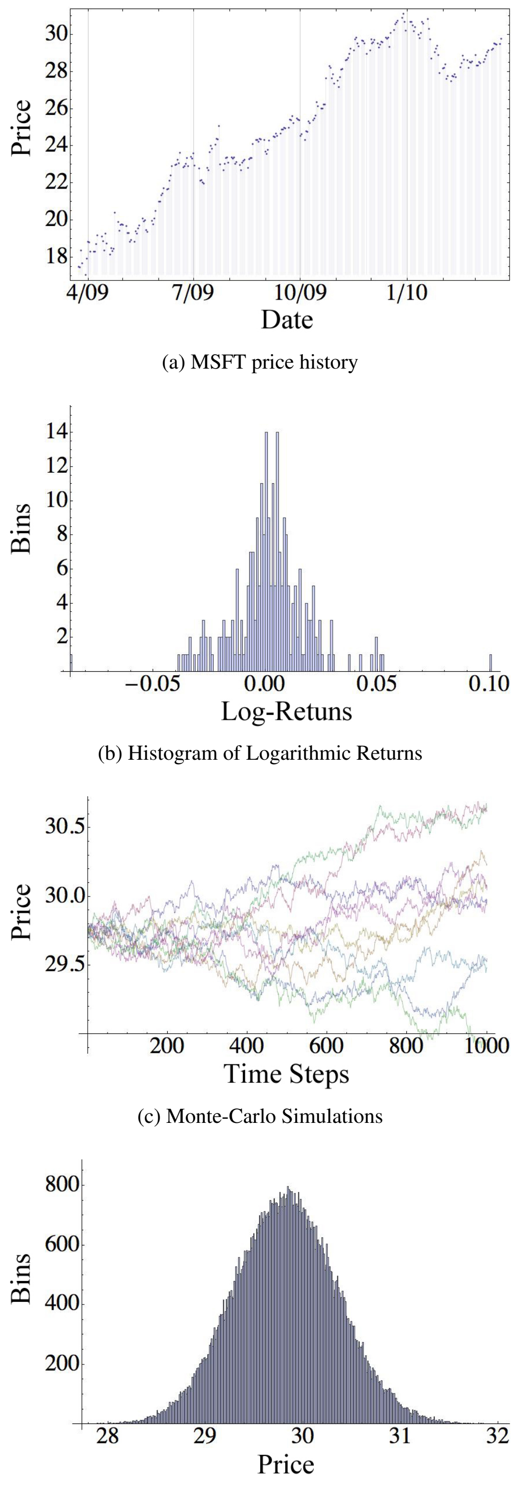
A practical requirement is that closing prices are reported consistently across publications. For example, in the Scantegrity II municipal election, both closing prices and closing volumes (the number of shares traded that day) were used. Auditor Ben Adida reports that the volumes that he accessed differed slightly from those used to generate the challenge, which could be due to differences in rounding, inconsistent reporting, or after market trades.[7] Since the data is being used to generate relatively small challenges, a malicious election authority could slightly perturb the volumes from their actual values until a suitable challenge is generated that hides any fraud. Even though this value differs from the reported values, it is indistinguishable from the scenario where the volumes were changed by the publisher.

For this reason, we recommend that only closing prices are used and not volumes. As the value of options and derivatives depend on the exact closing price of a stock, infrastructure for publishing a uniform closing price is in place. The official closing price is algorithmically determined (*e.g.,* by a closing cross) in a transparent way and then multicast by the Consolidated Tape Association (CTA), typically 15 minutes after the close of the markets. It clearly indicates which trades are considered after-market. Some newspapers or financial data sources may provide a “closing price” that is adjusted by after-market trades—these should not be used. A third party publisher may also make a mistake. We recommend that election officials (or the beacon service provider) check the closing prices from a few sources for consensus before generating the challenge, assuming they do not have direct access to the CTA multicast.

4 Entropy Estimates

In this section, we use the Black-Scholes model to estimate the entropy in a closing price. We illustrate the process by using the stock Microsoft (MSFT), which is included in the portfolio of stocks that we are ultimately interested in—the Dow Jones industrial average.

[7] B. Adida. Takoma Park: Meeting 2. `http://benlog.com/articles/2009/11/02/takoma-park-meeting-2/`.



(a) MSFT price history

(b) Histogram of Logarithmic Returns

(c) Monte-Carlo Simulations

(d) Histogram of Final Price

Figure 1: Estimating the entropy for MSFT. We (a) import one year of closing prices, (b) measure the relative price changes using logarithmic returns, extract estimates for drift and diffusion parameters and use these to (c) simulate the price over the next day, and (d) compute a histogram of simulated closing prices.

4.1 Historical Drift and Diffusion

Figure 1a shows the closing prices of Microsoft from March 23, 2009 at \$17.95 until March 23, 2010 at \$29.88. Let this series be $\{S_0, S_1, S_2, \dots S_T\}$. In this case, $T = 251$ given the number of trading days in the provided interval. From this data series, we can calculate the relative price changes using

$$R_i \quad = \quad \ln\left(\frac{S_{i+1}}{S_i}\right),\ 0 \le i \le T-1. \tag{1}$$

R_i is called the log (or continuously compounded) return. It is positive for relative increases in the stock price and negative for decreases. A histogram of R_i values for Microsoft is shown in Figure 1b with bin size 0.005. The distribution of R_i for this example is roughly normal, and any deviation from a normal distribution, as we will now derive, is evidence against the Black-Scholes assumption of asset prices following geometric Brownian motion.

With GBM, asset prices are modelled as

$$\mathrm{d}S_t \quad = \quad \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t. \tag{2}$$

With a logarithmic transform of S_t and application of Ito's lemma, GBM can be found to have the following analytic solution:[8]

$$S_t \quad = \quad S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right). \tag{3}$$

Due to the W_t term, this solution is a process. Thus if we are given a set of (S_0, S_t) pairs, we can estimate the values μ and σ to fit this process.

Let Δt be the sampling interval relative to the measured period. For example, if we sample daily prices and want the annualized distribution, Δt would be 251. In our case, we sample daily values to estimate the distribution over the same time-period: one day. Thus we use $\Delta t = 1$. From equation 3, the distribution should be

$$R_i \quad \sim \quad \mathrm{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right). \tag{4}$$

By taking our sampled data R_i and computing the standard deviation, we have an estimate for the daily diffusion σ. In finance, this is called the historic volatility (although not all volatility measures are calculated the same way). Next we find the mean of R_i, and estimate the drift term μ as: $\text{mean}(R_i) + \sigma^2/2$. For the MSFT data, we find the daily drift and diffusion estimators to be $\mu = 0.23\%$ and $\sigma = 1.77\%$ per day.

[8] Argument omitted for brevity. See an introductory computational finance textbook, *e.g.,* [38].

A first note on bias. Recall that the Black-Scholes model assumes that dividends are not paid out. This is not true for historic data. For the example, the MSFT data includes four dividend payments of \$0.13 each. This causes the closing price to be adjusted downward. Thus equation 1 should differentiate between pre- and post-adjusted prices: denominator S_i should be the post-adjusted price for time i, while numerator S_{i+1} should be the pre-adjusted (raw) price for time $i+1$. We found the difference to be insignificant due to dividends being infrequent and small, and thus ignored dividends.

4.2 Monte-Carlo Simulation

With estimates for μ and σ, we next consider the distribution of outcomes over the time-period, τ, for which we want to harvest entropy. To generate this distribution, we use a Monte-Carlo simulation of the process in equation 3. We discretize τ, which is one day for the MSFT example, into m equal-sized steps of size $\Delta\tau$. Equation 3 can then be simulated as Algorithm 1.

Algorithm 1: A Monte-Carlo trial for simulating asset price movements.

```
1 for $(0 \leq j < m)$ do
2     $t \leftarrow T + j \cdot \Delta\tau$
3     $Z_t \leftarrow_r \mathrm{N}(0,1)$
4     $S_{t+1} \leftarrow S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta\tau + \sigma Z_t\right)$
```

We begin at S_T, the last observed price (\$29.88 for MSFT), and map a possible trajectory to $S_{T+\tau}$ by stepping through Algorithm 1. Line 2 simply keeps the timestep notation tidy. Line 3 indicates a random variable drawn from a standard normal distribution. This variable is used in the next line, in conjunction with the μ and σ parameters to step the price forward one interval. We then repeat the algorithm N times to generate many independent possible outcomes for $S_{T+\tau}$. Figure 1c illustrates the first 10 trajectories, while Figure 1d shows a histogram of outcomes for 100 000 simulations and a bin size of one cent.

A second note on bias. Generally, Monte-Carlo is subject to three types of error. There is the discretization error due to modelling a continuously random process with m intervals. In our case, the exact solution in line 3 of Algorithm 1 is unbiased by discretization error because it is multiplicative. In models other than GBM, the only known solution may be an additive approximation instead (for example, GBM itself could alternatively be approximated by $S_{t+1} = S_t + S_t\mu\Delta\tau + S_t\sigma Z_t$ which is called Euler time-stepping). In this latter case, the error is order $O(m^{-1})$. The second source of bias is using N trials to estimate some value. In computational finance, the value of interest is the mean of the N outcomes. With both types of error, we reach the often stated total error of Monte-Carlo methods in computational finance: $O(\max(m^{-1}, N^{-0.5}))$ [38]. However in our case, we have an exact solution that eliminates the first term, and we are interested in the entropy of the distribution of N trials, not the mean, resulting in a different bias for the second term. We will discuss how N influences this bias in the next section after we have defined entropy.

The third possible source of bias is the statistical properties of the random number generator used for line 2 of Algorithm 1. We used the default generator in *Mathematica*, which we believe is more than sufficient for Monte Carlo simulations. It creates a seed from sessional information, uses cellular automata to expand the seed into pseudorandom bits, and Box-Muller to transform these into a normally distributed number.[9] We also experimented with a Mersenne twister, sometimes recommended for use in computational finance (*e.g.,* [38]), and it made no significant difference.[10]

Why use Monte Carlo? Given that we have an exact solution for GBM, in equation 3, we could eliminate the discretization step and generate $S_{T+\tau}$ values in a single step. Instead, we use time-stepping to create an approach that is easily replicable if one were to swap GBM for a different financial model where an exact solution is not known—*i.e.,* mean reversion or jump-diffusion models. Our aim is to assist the interested reader in studying different models.

4.3 Entropy Estimation

We now consider how much entropy is provided in a stock price over the course of a day. We first estimate the Shannon entropy, which provides an average-case measure of unpredictability, as this has the most intuitive appeal and highlights parameter dependence well. We later will consider min-entropy, which provides a worst-case measure. The Shannon entropy of discrete random variable X with probability mass function $p(x)$ is defined as

$$\mathrm{H}(X) \quad = \quad -\sum_x p(x) \log_2 p(x). \qquad (5)$$

Given the results from the Monte Carlo simulation, we have a list of N possible outcomes for the price of our

[9] `http://reference.wolfram.com/mathematica/tutorial/RandomNumberGeneration.html`

[10] Linear feedback shift registers, which Mersenne twisters are based on, are a specific type of cellular automata with a "spiral" boundary condition. See [40, pp. 1088].

asset. Denote these outcomes $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$, where P_i is $S_{T+\tau}$ for the i^{th} Monte Carlo trial. We round each outcome to the nearest cent and place it in a set of bins for each unique price. Since many trials will yield the same price once rounded, let $\hat{N} \leq N$ be the number of unique outcomes observed (*i.e.,* non-empty bins). Define $\hat{\mathbf{P}} = \langle \hat{p}_j, \hat{P}_j \rangle$, for $1 \leq j \leq \hat{N}$, as the set of pairs where $\hat{P}_j$ is a unique price in $\mathcal{P}$ and $\hat{p}_j$ is the number of times it appears.

To estimate the Shannon entropy in our set $\mathcal{P}$, we compute

$$\mathrm{H}(\mathcal{P}) \quad = \quad -\sum_{j=1}^{\hat{N}} \frac{\hat{p}_j}{N} \log_2\left(\frac{\hat{p}_j}{N}\right). \qquad (6)$$

This type of estimator is known as a maximum likelihood, naive, or plug-in estimator [34]. It works by distributing the random variable into bins and estimating $p(x)$ by dividing the number of outcomes in each bin by the total number of outcomes. The goal of this paper is to estimate the entropy of a closing price, rounded to the nearest cent, which is a discrete random variable. So we use a bin size of one cent.

A third note on bias. Maximum likelihood estimators (MLE) for Shannon entropy are biased. As with any random sampling, some bins may have more values than they theoretically should and others less and this tends to average out as N increases. However for entropy estimation, an empty bin cannot be included in Equation 6 because $0 \cdot \log(0)$ is undefined. Instead, empty bins are dropped from the estimate even if theoretically they should be non-empty. This leads to a negative bias for MLE and the entropy is lower than it should be.

In our case, we want a conservative estimate of entropy and so negative biases of this sort are not so troubling. However the bias can be corrected if we can provide an estimate of how many bins have non-zero probability (relative to the number of samples). To estimate this value, we take the full range of $\min(\mathcal{P})$ to $\max(\mathcal{P})$. Let this be $\hat{M}$ bins. The Miller-Madow bias [33] of an MLE is given as

$$B \quad = \quad \frac{\hat{M} - 1}{2N}. \qquad (7)$$

As stated, it is a negative bias and so the adjusted entropy is: $H_B(\mathcal{P}) = H(\mathcal{P}) + B$. Other adjustments exist in the literature [34]. We selected Miller-Madow because it is computationally easy to compute for large values of N and appropriate when $N > \hat{M}$.[11] For the MSFT data that we have been using to illustrate each step, we used $N = 100\,000$ and found $\hat{N} = 397$ and $\hat{M} = 447$. The MLE Shannon entropy is 7.764 bits and the Miller-Madow bias is 0.002, which is relatively small.

Min-Entropy. While Shannon entropy provides an estimate of the average entropy in a stock price, a worst-case estimate is needed if we want to extract the randomness out of the price. Given a distribution $\mathcal{X}$, the min-entropy, $H_\infty(\mathcal{X})$, is defined as

$$\mathrm{H}_\infty(\mathcal{X}) \quad = \quad -\log_2(\max_x(p(x))). \qquad (8)$$

In other words, for any possible outcome x, $p(x) \leq 2^{\mathrm{H}_\infty(\mathcal{X})}$. If $\mathcal{X}$ is uniformly random, the Shannon entropy and min-entropy are equivalent. Otherwise, min-entropy is strictly less. Since $H_\infty(\mathcal{X})$ is ultimately computed from one probability $p(x)$ and this value will be non-zero if the entropy is non-zero, the bias from empty bins on Shannon entropy does not apply to the notion of min-entropy.

We use the Shannon entropy estimates to examine the effect of drift, diffusion, initial price, and elapsed-time in the next subsection. We use the min-entropy estimate when we want to configure a random extractor to produce a short, near uniform-random bit-string from the much longer set of prices.

4.4 Experimental Results

In the Scantegrity II municipal election, a portfolio of 30 stocks was used.[12] These stocks were the companies that make up the Dow Jones industrial average (DJIA) — an important financial benchmark. Table 1 shows the data that we collected and our estimates for each of these stocks following the approach outlined for the MSFT example. The prices were observed from March 23, 2009 to March 23, 2010 (S_0 to S_T), and these were used to estimate the daily drift and diffusion rates: μ and σ. With these parameters, we simulated the path from the closing price on March 23, 2010, S_T, forward one day in time using a Monte Carlo simulation with $100\,000$ trials. The number of unique prices (to the nearest cent) in our simulation, $\hat{N}$, is used to generate an MLE-estimate for the Shannon entropy: $H(\mathcal{P})$. The estimated number of expected non-empty bins, $\hat{M}$, is used to estimate the Miller-Madow bias: B. These are combined to generate our adjusted estimate of Shannon entropy: $H_B(\mathcal{P})$. Finally, we also provide our estimate of the min-entropy: $H_\infty(\mathcal{P})$. Pfizer (PFE) had the lowest entropy: 6.83 and 6.10 bits for Shannon and min-entropy respectively, while Caterpillar (CAT) had the highest at 9.46 and 8.69 bits respectively.

[11] More precisely, if $N/\hat{M}$ diverges to ∞ as N grows, then $H_B(\mathcal{P})$ will converge on the correct result.

[12] B. Adida. Takoma Park: Meeting 2. `http://benlog.com/articles/2009/11/02/takoma-park-meeting-2/`

Stock	S_T	μ	σ	$\hat{N}$	$\hat{M}$	$H(\mathcal{P})$	B	$H_B(\mathcal{P})$	$H_\infty(\mathcal{P})$
AA	14.50	0.00338956	0.0354406	386	440	7.72544	0.0022	7.73	6.99
AXP	41.24	0.00512444	0.0365912	1071	1305	9.2823	0.006525	9.29	8.50
BA	72.18	0.00313975	0.0219116	1112	1406	9.34279	0.00703	9.35	8.57
BAC	17.13	0.00455058	0.0468486	588	699	8.37453	0.003495	8.38	7.62
CAT	62.41	0.00352696	0.027272	1173	1540	9.4536	0.0077	9.46	8.69
CSCO	26.64	0.00200486	0.0167037	347	396	7.52981	0.00198	7.53	6.79
CVX	74.77	0.000565046	0.0136131	730	844	8.70798	0.00422	8.71	7.95
DD	38.31	0.0025213	0.0219181	603	751	8.43244	0.003755	8.44	7.69
DIS	34.01	0.00271648	0.0199576	506	596	8.13347	0.00298	8.14	7.39
GE	18.33	0.00264998	0.0239698	335	391	7.50284	0.001955	7.50	6.76
HD	32.59	0.00166033	0.0161739	404	475	7.76791	0.002375	7.77	7.03
HPQ	53.15	0.00234904	0.015783	615	758	8.43501	0.00379	8.44	7.69
IBM	129.37	0.00124652	0.0124436	1121	1460	9.36931	0.0073	9.38	8.60
INTC	22.67	0.0019257	0.0176758	302	352	7.37295	0.00176	7.37	6.63
JNJ	65.36	0.00101973	0.00811278	406	472	7.7723	0.00236	7.77	7.03
JPM	44.58	0.00261719	0.0318482	992	1190	9.18918	0.00595	9.20	8.43
KFT	30.49	0.00134673	0.0129888	314	333	7.35464	0.001665	7.36	6.62
KO	55.30	0.0010976	0.0111199	460	570	7.98678	0.00285	7.99	7.21
MCD	67.35	0.00111279	0.0113681	569	732	8.3043	0.00366	8.31	7.55
MMM	82.35	0.00235099	0.0148201	854	1075	8.97369	0.005375	8.98	8.22
MRK	38.50	0.00162879	0.0166847	486	554	8.05935	0.00277	8.06	7.29
MSFT	29.88	0.0022737	0.0176583	394	449	7.76265	0.002245	7.76	7.04
PFE	17.54	0.00120496	0.01571	216	243	6.82701	0.001215	6.83	6.10
PG	64.53	0.00146004	0.0125241	587	703	8.37914	0.003515	8.38	7.64
T	26.55	0.000357228	0.0121909	251	289	7.05851	0.001445	7.06	6.31
TRV	53.90	0.00154645	0.0188065	734	926	8.7059	0.00463	8.71	7.96
UTX	73.09	0.00232501	0.0159515	835	1015	8.9016	0.005075	8.91	8.14
VZ	30.98	0.000367966	0.0117435	279	320	7.22926	0.0016	7.23	6.48
WMT	55.89	0.000497465	0.010295	431	512	7.89168	0.00256	7.89	7.16
XOM	66.95	0.0000317968	0.012391	604	752	8.41962	0.00376	8.42	7.65

Table 1: The Dow Jones portfolio of 30 stocks. For the Monte-Carlo parameters, we show: initial price (S_T), historic diffusion parameter (μ), and historic drift parameter (σ). For the Shannon entropy estimate, we show: the number of unique prices in the simulation ($\hat{N}$), the estimated number of non-empty bins ($\hat{M}$), the Shannon entropy estimate based on observed prices ($H(\mathcal{P})$), the Miller-Madow bias (B), and the bias-adjusted estimate ($H_B(\mathcal{P})$). For the min-entropy estimate, we show the estimate based on observed prices ($H_\infty(\mathcal{P})$).

	Drift		Diffusion		Initial Price		Time	
	μ	$H(\mathcal{P})$	σ	$H(\mathcal{P})$	S_T	$H(\mathcal{P})$	τ	$H(\mathcal{P})$
Min	0.003%	6.81	0.81%	8.53	\$14.50	7.33	0.5 day	8.03
Mean	0.195%	8.54	1.87%	8.54	\$ 48.02	8.54	1 day	8.54
Max	0.513%	9.95	4.68%	8.54	\$ 129.37	9.85	2 days	9.03

Table 2: This table shows the result on the entropy of the closing price if each parameter is independently changed from its average value across the DJIA to its minimum and maximum.

We were also interested in the individual effect of drift, diffusion, opening price, and time-period on the entropy of a stock. We created a mythical stock with the mean value for each of these parameters, calculated from the DJIA data. The stock had $\mu = 0.195\%$, $\sigma = 1.87\%$, and $S_T = \$48.02$ and was simulated over one day. For each parameter, we individually varied it to the minimum and maximum observed value for this parameter in the DJIA data and estimated the resulting Shannon entropy. We also varied the timeframe from half a day to two days. The results are provided in Table 2. The entropy was sensitive to the observed spread in both μ and S_T but largely invariant to changes in σ. In all cases, an increase in the parameters resulted in an increase in the entropy.

4.5 Portfolio Entropy

We have shown how to estimate the entropy of individual stocks. But how much entropy is in a collection of stock prices? From Figure 1, it may be tempting to sum the entropy estimates for all the stocks and use this as an estimate of the total entropy in the portfolio. This approach works only if the stocks are uncorrelated with each other. In reality, stocks typically display varying degrees of correlation with other stocks from, for example, the same business sector, same country, or when traded on the same index. This means that the portfolio entropy is less than the sum of the entropy of the individual stocks due to mutual information between subsets of the stocks.

We selected the two stocks with the highest correlation and estimated the mutual information between them. Pairwise throughout the portfolio, the highest correlated stock pairs are Cheveron and Exxon (0.82 over one year) which are both large oil and gas producers. Next are JP Morgan and Bank of America (0.78) which are both large banks. The mean correlation was 0.42.

Taking Cheveron and Exxon (CVX and XOM), we generated correlated Monte Carlo paths [38]. To do this, recall Algorithm 1. We perform this algorithm for CVX. Denote the value of Z_t in line 3 for CVX as Z_t^C. For XOM, our second stock, we will run almost the same algorithm; we replace line 4. First denote the output at line 3 as Z_t^X and let the correlation between CVX and XOM be ρ. Then the replacement for line 4 is

$$\begin{aligned} S_{t+1}^X \leftarrow S_t^X \exp\Bigg(\left(\mu - \frac{\sigma^2}{2}\right)\Delta\tau \\ +\sigma\left(\rho Z_t^C + \sqrt{1-\rho^2} Z_t^X\right)\Bigg). \end{aligned} \tag{9}$$

In other words, Z_t^X is used as a joint random variable shared between both stocks.

Using this method, we ran 10 000 000 trials to estimate the joint Shannon entropy between Exxon and Cheveron. This was the largest simulation that was computationally feasible for us. Recall from Table 1 that the number of unique prices observed for Cheveron and Exxon were respectively 730 and 604. That means the number of unique price pairs is on the order of 730*604. Thus our simulation of 10M trials was only one order of magnitude greater than the number of observed events and our result is a quite sparse histogram from which the estimates will be sufficiently biased. We found that the joint entropy was 15.96 bits compared to the sum of their individual entropies: 16.90 bits. That means the mutual information is at most 0.94 bits. Again, we did not adjust for bias (Miller-Madow is best when the trials are much larger than the outcomes) and so the mutual information is likely less than this. For min-entropy, the result is 1.04 bits.

We leave a rigorous analysis of mutual information in the entire Dow Jones index for future work. As mentioned, computing the joint entropy between two stocks is very difficult as it is: the number of bins squares, as does the number of trials needed to create a suitably dense histogram. Methods exist for estimating bias when the trials are less than the number of bins [34] but it is not obvious how to extend these estimators to more than two random variables. Computing the joint entropy between 30 stocks does not seem computationally feasible, even if the trials could be less than the resulting bins by a polynomial factor.

We can provide a very crude estimate by making a generous assumption. Recall the chain rule for joint entropy is as follows:

$$H(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n) = \sum_{i=1}^{n} H(\mathcal{P}_i | \mathcal{P}_{i-1}, \dots, \mathcal{P}_1). \tag{10}$$

We could estimate the $H(\mathcal{P}_i|\mathcal{P}_{i-1}, \dots)$ term for each stock as: $H(\mathcal{P}_i) - \max_{(i,j\neq i)}\left(I(\mathcal{P}_i, \mathcal{P}_j)\right)$. This would hold if the worst-case mutual information between any two stocks in the portfolio was less than the mutual information a given stock has with the rest of the portfolio. In other words, any mutual information Cheveron shared with a stock other than Exxon (because of similarities in sectors, country, exchange, *etc.*) would already be accounted for in the mutual information it shares with Exxon. This is crude and the estimate should be treated only as a ballpark figure. For the DJIA under this assumption, the Shannon entropy is 218 bits and min-entropy is 192.

5 Beacon Implementation

We have estimated that the closing price for a single stock in the DJIA provides between 6 and 9 bits of entropy, and we have shown that adding additional stocks increases the entropy. In this section, we consider how to convert a list of closing prices into a form that is useful for general cryptographic protocols. Since the field of cryptography has conventions in notation that sometimes conflict with conventions in computational finance or information theory, we will, in a small number of cases, redefine variables in this section.

Let $\mathcal{P}_i$ be a list of closing prices from our portfolio on day i. We encode the prices as integers with a fixed, sufficient number of digits and concatenate them: *i.e.,* $\{14.34, 41.08, \ldots\} \rightarrow 001434 \| 004108 \| \ldots$. Let the bit-length $|\mathcal{P}_i| = n$. If the prices in the list are, for example, the 30 stocks of the DJIA, encoding each price with 6 digits will produce a 180 digit or 598 bit string. However from our simulations in the previous section, we estimate that there would be only 218 bits of entropy in this 598 bit string, and 192 extractable bits.[13] Because the randomness is not concentrated, this semi-random string is not a suitable for seeding selection algorithms or pseudorandom generators, or as a cryptographic challenge. In this section, we provide a protocol that, among other things, takes as input a long, semi-random string of stock prices, and output a shorter string of near-uniform random bits.

The stock market prices can be used in at least one of two ways: it can be sampled directly by any party requiring a random challenge, or, alternatively, a neutral third party can regularly sample the source, produce a random bit-string in some useful form, and publish it for quick reference by any party requiring an unpredictable challenge (or seed, common reference string, nonce, *etc.*). We call such a publisher a *beacon service provider* (BSP) and we will refer to the random values it produces as $seeds$. In general, the BSP will be agnostic of what the seeds are being used for.

A BSP can provide a few benefits: if an election (or other protocol) has a low risk of fraud, the fact the entity claims a seed is random may be trustworthy enough. In higher risk protocols, the derivation of the seed can be independently verified and the BSP is not trusted at all. Since the BSP is regularly publishing a new seed, it can update in a recursive fashion, mixing the new closing prices with the previous seed. This makes the output dependent on the prices that preceded it. Finally, as an incentive to provide a beacon service, we can, with care, also allow the BSP to influence the seed with its own randomness. This also provides a marginal hedge for everyone else: even if the closing prices are fully manipulated, the attacker will still have to also collude with the BSP to fix the seed.

5.1 Definitions

Randomness Extractor. Briefly, $\texttt{Ext}_k$ is a function: $\{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^m$. It takes an n-bit input of sufficient entropy and a d-bit key k and returns an m-bit output of high entropy where $m < n$. Here sufficient entropy means the min-entropy of $\{0,1\}^n$ is at least m bits (we will generally require $2m$) and high entropy means that the statistical distance between the distribution on $\{0,1\}^m$ and m uniform random bits is ϵ, where ϵ is a negligible function in m. We only consider the case where $d = m$. For a formal definition, see [17].

CBC-MAC. Briefly, CBC-MAC is a mode of operation for a block cipher based on cipher block chaining (CBC) with an initialization vector of zero, but it returns only the final block. It is suitable for creating message authentication codes (MAC). Dodis *et al.* show that CBC-MAC is an extractor if the block cipher is an ideal random permutation, the plaintext has $2m$ bits of min-entropy for an m-bit output, and the key is uniformly random. For the full details (including an upper bound on the input size), see [17].

Pseudorandom Generator. Briefly, G is a function: $\{0,1\}^m \rightarrow \{0,1\}^{l(m)}$. It takes an m-bit seed and returns a polynomially-bounded number of bits $l(m)$ typically greater than m. The distribution on $\{0,1\}^{l(m)}$ is ϵ-close to uniform random. For a formal definition, see [6].

Robust Pseudorandom Generators. Barak and Halevi provide a construction for a robust pseudorandom generator [6]. A robust PRG uses a standard PRG, G, as well as maintaining some evolving internal state s_i that can be referenced during operation. G is cryptographically secure: the internal state cannot be inferred from the output by an adversary bound to probabilistic polynomial time (PPT). The construction separates the task of updating the state from the task of generating pseudorandom bits. We only reference the former function: `refresh`.

$$\begin{aligned} \texttt{refresh}: \quad & \{0,1\}^m \times \{0,1\}^n \rightarrow \{0,1\}^m \\ & s_i \leftarrow \texttt{G}(s_{i-1} \oplus \texttt{Ext}_k(\mathcal{P}_i)). \end{aligned}$$

This function is of interest because it could be used for updating the seeds of a beacon service, where the seed is substituted for the state. A robust PRG affords a number of functions to an adversary. Of interest, `badRefresh` lets the adversary refresh with a chosen $\mathcal{P}_i$, and `setState` allows the adversary to learn s_i and replace it with a chosen s_{i+1}. In the face of these attacks, a robust PRG provides three properties: forward secrecy

[13] Even if we did not prepend each price with as many leading zeros, the entropy would still be less than the length of the string. This is because the most significant digits in the price is much less likely to change than the least significant digit (*i.e.,* the one cent digit).

(not needed for a beacon service since past seeds are public), break-in recovery, and resilience. Briefly, break-in recovery means that if an adversary learns s_i for round i, s_{i+1} is still unpredictable if `refresh` is run with a sufficiently random $\mathcal{P}_{i+1}$. Resilience means that if an adversary controls $\mathcal{P}_{i+1}$ but does not know s_i, s_{i+1} is unpredictable.

Bilinear Groups. Let $\mathbb{G}$ and $\mathbb{G}_T$ be cyclic groups of order q, where q is a large prime. Let $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$ be a bilinear, non-degenerate, and efficient mapping. That is, $\forall g_1, g_2 \in \mathbb{G}$ and $\forall a_1, a_2 \in \mathbb{Z}_q$: $e(g_1^{a_1}, g_2^{a_2}) = e(g_1, g_2)^{a_1 \cdot a_2}$ where e is computable in polynomial time and $e(g, g)$ generates $\mathbb{G}_T$ for some $g \in \mathbb{G}$. Finally, assume the q-decisional bilinear Diffie-Hellman inversion assumption holds for the group (definition omitted for brevity – see [9]).

Verifiable Unpredictable Function. Briefly, a VUF is a tuple of four functions. `GenK` returns secret key sk and public key pk. $\texttt{Eval}_{sk}(x)$ takes input x and returns a unique and unpredictable y (we omit defining the domain and range, as they vary by construction). $\texttt{Proof}_{sk}(x, y)$ returns proof $\pi_{x,y}$ that y is the output on x. $\texttt{Verify}_{pk}(x, y, \pi_{x,y})$ returns 1 iff proof is correct. A VUF is a weaker primitive than a verifiable random function (VRF). Given x, a VUF guarantees y cannot be guessed with non-negligible advantage, while a VRF offers the stronger guarantee that y cannot be distinguished from a random element with non-negligible probability. A VUF is sufficient for us because we know the entropy in y and will use an extractor to convert it into the required form. For a formal definition of both, see [18].

Dodis-Yampolskiy VUF. Dodis and Yampolskiy propose the following VUF, defined over a bilinear group: $\{\mathbb{G}, \mathbb{G}_T, g \in \mathbb{G}, q\}$ [18]. `GenK` publishes the group, randomly generates secret key $sk \leftarrow_r Z_q^*$ and publishes public key $pk = g^{sk}$. $\texttt{Eval}_{sk}(x)$ takes $x \in Z_q^*$ and returns y, which in practice is a point on an elliptic curve. $\texttt{Eval}_{sk}(x)$ is defined as $y \leftarrow g^{\frac{1}{x+sk}}$. The proof is embedded in y; $\texttt{Proof}_{sk}(x, y)$ simply returns $\{x, y\}$. $\texttt{Verify}_{pk}(x, y, \pi_{x,y})$ is defined as $e(pk \cdot g^x, y) \overset{?}{=} e(g, g)$.

5.2 Security Properties

The central function of our BSP protocol updates the seed at interval i with new randomness $\mathcal{P}_i$. We define it with the recurrence: $s_i \leftarrow \texttt{Update}(s_{i-1}, s_{i-2}, \mathcal{P}_i)$, where s_i is the seed at interval i and s_{i-1} and s_{i-2} are the two preceding seeds. An update function does not need to use *two* previous seeds—it could be more or less. The seeds are akin to the internal state of a robust PRG, except that they are made public at each interval. We assume the BSP has a secret key. We have several properties we want from our protocol.

- **Verifiable.** A polynomial time verifier should be convinced that `Update` was computed from $\mathcal{P}_i$, s_{i-1}, and s_{i-2} as specified in the protocol.
- **Statistically Random.** The statistical difference between s_i and a uniform distribution of the same length should be ϵ-close.
- **Unpredictable.** The advantage of a PPT-bound adversary in predicting s_i without $\mathcal{P}_i$ is negligible, even if the adversary knows s_{i-1}, s_{i-2} and the BSP's secret key.
- **Parisian.** The advantage of a PPT-bound adversary in computing the exact value of $\mathcal{P}_i$ needed to output a chosen s_i is negligible if s_i is chosen before knowing $\mathcal{P}_{i-1}$ and s_{i-1} and s_{i-2}, even is she knows the BSP's secret key.
- **Partially Distributed.** The advantage of a PPT-bound adversary in predicting s_i is negligible if she knows s_{i-1} and s_{i-2}, chooses $\mathcal{P}_i$, but does not know the BSP's secret key.

With the exception of the Parisian property, the motivation for these is hopefully intuitive and not in need of further explanation. We will illustrate the last property with a concrete example. Suppose on the Tuesday night of a given week, election officials post the results of an election and the post-election challenges are to be generated on the Friday of that week after the markets close. On Thursday night, the adversary knows the seed from Thursday and let us assume she can fully manipulate Friday's closing prices. Further, assume that she has a particular seed that she wants produced to ensure some fraud committed on Tuesday escapes detection. It should be computationally infeasible for the adversary to determine the closing prices she needs to set in order to generate, from the existing seeds, this target seed she wants. The only way the adversary can produce the desired seed on Friday is to manipulate the prices every day: Wednesday, Thursday, and Friday. We name this "parisian" after the Parisian options we discussed earlier, which required a barrier event to occur on multiple days.

5.3 Protocol

Variants. In this section, we define a protocol that can be used by a BSP to implement the `Update` function described in the previous section. The stock market randomness, $\mathcal{P}_i$, is n bits and contains $2m$ bits of min-entropy. We reference two extractors with different output sizes: $\texttt{Ext}_k$ has an m-bit key and produces an m-bit

	Variant	Verifiable	Unpredictable	Statistically Random	Parisian	Partially Distributed	Output Length ($\lvert s_i \rvert$)
1.	$s_i = \mathcal{H}(\mathcal{P}_i)$	x	x				$\{0,1\}^h$
2.	$s_i = \texttt{Ext}_k(\mathcal{P}_i)$	x	x	x			$\{0,1\}^m$
3.	$s_i = \mathcal{F}_{\text{owf}}(s_{i-1} \oplus \texttt{Ext}_k(\mathcal{P}_i))$	x	x	x	x		$\{0,1\}^m$
4.	$s_i = \texttt{Ext}'_{k'}(\texttt{Eval}_{sk}((s_{i-1} \| s_{i-2}) \oplus \texttt{Ext}_k(\mathcal{P}_i)))$	x	x	x	x	x	$\{0,1\}^l$

Table 3: A sequence of intermediate protocols, and which properties they achieve, leading to our suggested protocol. Assuming $\mathcal{P}_i$ has $2m$ bits of min-entropy, the output length is provided. Length h depends on the hash function used, while $m = 2l$ (*e.g.,* h=160, m=256 and l=128).

output, while $\texttt{Ext}'_{k'}$ has an l-bit key and l-bit output. We consider $m = 2l$.

Table 3 shows some related constructions. Perhaps the simplest approach is to publish an h-bit hash of the list of closing prices each day: Variant 1. Before $\mathcal{P}_i$ is known, the hash of $\mathcal{P}_i$ will trivially be unknown as well, and once $\mathcal{P}_i$ is known, anyone can verify the output is correct by recomputing the hash from their source for the closing prices.

The problem with using a fixed hash function is that it does not guarantee a statistically random output. Dodis *et al.* show that even if a hash function is modelled with an ideal compression function (and Merkle-Damgaard chaining), it does not have good extraction properties. Instead, the hash needs to be taken from a family of hash functions (*i.e.,* a keyed hash) and even then, one must pay attention to the padding scheme used to ensure the final block has sufficient entropy [17]. With the use of a proper extractor, Variant 2 of the protocol produces a statistically random output. While extractors are keyed, the key, k, only needs to be uniformly random. Its value is not kept secret and the key can be reused.

Variant 2 is not Parisian because it is only functionally dependent on the most recent closing prices of the stocks. Let $\mathcal{F}_{\text{owf}}$ denote a one-way function. Variant 3 adds the Parisian property and is essentially the `refresh` function of a Barak-Halevi robust PRG. Recall `refresh` is defined as: $s_i \leftarrow \texttt{G}(s_{i-1} \oplus \texttt{Ext}_k(\mathcal{P}_i))$. `G` is used as: $\{0,1\}^m \rightarrow \{0,1\}^m$ (*i.e.,* there is no expansion of m) and PRGs are one-way. [14]

Variant 4 is our protocol. The main modification is to replace $\texttt{G}(\cdot)$ with the $\texttt{Eval}_{sk}(\cdot)$ function of a VUF, which allows the BSP to influence the seed with sk, while maintaining verifiability. This adds the partially distributed property. The other variants were verified by recomputing the value. In this case, the BSP produces a correctness proof for $\texttt{Eval}_{sk}(\cdot)$ that is checked. The properties of the VUF also imply that without knowledge of sk, the output of $\texttt{Eval}_{sk}(\cdot)$ is unpredictable if the input is unknown.

VUF Details. A variety of VUFs and VRFs exist. We use the Dodis-Yampolskiy VUF [18] for its simplicity, efficiency, and the fact that its proof is non-interactive.[15] However since it is based on bilinear pairings, we require some encoding. Recall the domain of `Eval` is an exponent and the range is a point on an elliptic curve. Let Φ_1: $\{0,1\}^m \rightarrow \mathbb{Z}_q^*$ be an encoding function that is entropy preserving (*i.e.,* injective). When $q > 2^m$, this encoding is the trivial one. Let Φ_2 be a mapping from an element in $\mathbb{G}_T \rightarrow \{0,1\}^m$. This encoding (or extraction) is more difficult, and it is specific to the type of elliptic curve used. For a random point on an ordinary curve defined over $\mathbb{F}_{2^{2l}}$, an l-bit string can be extracted [23]. For a supersingular curve (which offer fast pairing operations), we are not aware of any specific extractor. Instead, we simply concatenate the coordinates together and use a randomness extractor, $\texttt{Ext}'_{k'}$, which produces an l-bit output.

Extractor Details. In deciding on how to implement $\texttt{Ext}_k$, we consider two options.

- We could use an approach that is specific to the distribution we are drawing from: closing prices (for the first extraction). For example, we could consider a daily relative increase in price as heads and

[14] So why is `G` not a one-way function in the Barak-Halevi model? The answer is because we are only using one of two functions offered by the robust PRG model—the other function uses the same G as: $\{0,1\}^m \rightarrow \{0,1\}^{2m}$.

[15] Note that while Dodis-Yampolsky restrict the domain of their VUF, Camenisch *et al.* find it can admit the full range under a stronger security assumption (in the generic group model) than q-BDHI [14].

a decrease as tails. Due to the drift term, there is a slightly biased toward heads which can be corrected for with a Von Neumann extractor—see Footnote 1. However this approach produces less than a bit of entropy per closing price (and therefore does not guarantee even one bit of entropy each day the market closes) and since stocks are correlated, it is not clear how to use more than one stock.

- Dodis *et al.* investigate using a block cipher in CBC-MAC mode—the result quoted above. This construction has less "fine-print" than some of the alternatives they examine. If the plaintext has $2m$ bits of min-entropy and is L blocks long, it produces an m-bit output with a statistical distance, from a uniformly random m-bits, of $\mathcal{O}(L\ 2^{-m/2})$. A side-benefit is that it also has an "avalanche-effect" where a difference in a single bit of the input causes a random-looking difference in the output. We use this approach.

Key Derivation. The CBC-MAC extractor, or any other non-deterministic extractor, does have one issue however: how it is keyed. Extractor keys do not need to be secret (in fact, for verifiability, they cannot be), however they must be uniformly random. Again, we considered a few options.

- The BSP could choose a uniformly random key at the beginning of the protocol and use it throughout. However this is providing adversarial control over the key, and we cannot cite any strong results on what an adversary could do with this control.
- Assuming we can bootstrap the process, we will be generating good quality randomness at every iteration and so we could use the values of the previous seeds to refresh the key to the extractor. The results from the extraction literature assume a uniform random key and do not indicate if an ϵ-close random key is sufficient (or could be compensated for by increasing the entropy of the input).
- We could use the historic prices of a single stock and a Von Neumann extractor as described in the first bullet point of the preceding list. While we rejected this approach for the extractor itself, it works well for generating a key. Since there are no unpredictable or secrecy requirements on the key, we can use past prices instead of future prices. This gives us immediate access to enough bits to generate the two required extraction keys (k and k'). In addition, the key can be continually updated by shifting in new bits as time goes by. We give full details of this key derivation procedure in Appendix A.1 in the full version of the paper.[16]

Algorithm 2: Update Beacon

```
begin
    $x_i = (s_{i-1} \| s_{i-2}) \oplus \texttt{AES256-CBC-MAC}_{k_i}(\mathcal{P}_i)$
    $y_i = g^{\frac{1}{x_i + sk}}$
    $s_i = \texttt{AES128-CBC-MAC}_{k'_i}(y_i)$
end
Publish: $\{y_i, s_i\}$
```

Algorithm 3: Verify Beacon

```
begin
    $x_i = (s_{i-1} \| s_{i-2}) \oplus \texttt{AES256-CBC-MAC}_{k_i}(\mathcal{P}_i)$
    $e(pk \cdot g^{x_i}, y_i) \overset{?}{=} e(g, g)$
    $s_i \overset{?}{=} \texttt{AES128-CBC-MAC}_{k'_i}(y_i)$
end
```

Protocol. The BSP publishes pairing-friendly $\{\mathbb{G}, \mathbb{G}_T, g \in \mathbb{G}, q\}$. The BSP chooses sk randomly from Z_q^* and publishes $pk = g^{sk}$. The BSP generates the current extraction keys k_i and k'_i. The initial states s_{-1} and s_0 are set to zero. On day $i \geq 1$, BSP takes the closing stock prices, P_i, and executes Algorithm 2. To verify that s_i is a proper update to s_{i-1}, a verifier with $\{pk, k_i, k'_i, \mathcal{P}_i, y_i, s_i, s_{i-1}, s_{i-2}\}$ executes Algorithm 3. Note that the verifier does not need to verify that the seeds from s_{i-1} back in time to s_1 were themselves correctly formed. To be assured that s_i is a random beacon, it is sufficient to check it is correctly formed from the arbitrary values claimed to be s_{i-1} and s_{i-2}. This is because $\mathcal{P}_i$ fully refreshes the randomness of s_i.

Parameter Sizes. For the extractors, we need a block cipher and AES is the default candidate. However the choice of AES locks us into only using extractors with 128, 192, or 256 bit outputs. Our protocol requires two extractors and with each extraction, we lose half of the bits of entropy in the input. Therefore we start with 512 bits of min-entropy in $\mathcal{P}_i$ and use AES-256 to generate a value x_i with (ϵ-close to) 256 bits of min-entropy. These 256 bits are preserved in y_i and then a second extraction with AES-128 produces a value s_i with (ϵ-close to) 128 bits of min-entropy.

This approach depends on $\mathcal{P}_i$ having at least 512 bits of min-entropy. We estimated the DJIA only has 192 bits. However we have also shown that adding additional stocks increases the total min-entropy. We would like the portfolio to be easy to construct, and ideally iconic—for this reason, we recommend using the S&P 500. By extrapolating our results, the daily closing prices of these

[16] Available at: `http://eprint.iacr.org/2010/361`

500 stocks provides 512 bits of min-entropy with a large safety margin.[17] This will allow us to produce a fresh 128-bit random number every market day.

This approach also assumes we can encode the 256-bit number x_i into the subgroup of a pairing-friendly curve. Common curve sizes, however, work in subgroups of 160-bits and so this would require a custom implementation. If alternative sizes for the extractors are preferable, one could use a variable-length block cipher [4] instead of AES, which allows extractors of any size.[18]

5.4 Security Analysis (Abstract)

Due to space restrictions, we omit the analysis of security for our protocol; however, it is included in Appendix A.2-7 of the full version of this paper[19]. In the analysis, we prove that our protocol has the five properties we have specified. We note that some of the properties we want to prove are subsumed by the properties of a Barak-Halevi robust PRG. This includes directly: *statistically random* and *unpredictable* (unpredictable is equivalent to their notion of break-in recovery). *Parisian* is also implied but not explicit. However, we prove each of these independently for our variant, plus demonstrating *verifiability* and the *partially distributed* property.

6 Concluding Remarks

In this paper, we have shown that the closing prices for common stocks contain sufficient min-entropy for generating random challenges for use in elections or other cryptographic applications. This result, in combination with the ease with which stock prices can be verified, makes financial data an attractive source of randomness for cryptographic voting systems unable to use the Fiat-Shamir heuristic, and possibly for precinct selection in standard electronic voting. In addition to the entropy estimates, we have provided a provably secure protocol for implementing a beacon service. It is our hope that such a beacon service, whether using our protocol or a variant, becomes a reality.

Future Work. We list a few items for future work. First, estimating joint entropy between correlated stocks becomes infeasible as the number of stocks grows. Progress on making this estimation more efficient would be welcome. Second, some results concerning extractors built with standard cryptographic primitives would be useful: specific bounds (instead of asymptotic bounds), analysis of using inputs where the min-entropy is less than twice the min-entropy of the output, and analysis of extraction with a key that is only statistically-close to uniform random, instead of being exactly uniform random. A third item for future work would be alternative VUFs, in particular a scheme that works over the integers, to reduce the complexities of mapping from a finite-field over elliptic curves back to the integers. Ideally, if a VUF mapped m-bit integers to m-bit integers, then the second extractor would not be needed at all.

Acknowledgements. The authors thank the Punchscan and Scantegrity teams for many discussions on the use of financial data as beacons. In particular, we acknowledge Ron Rivest for the idea of using the DJIA as an entropy pool and for the protocol used in the Scantegrity II election in Takoma Park. The issue of inconsistent reporting of stock volumes was raised by Ben Adida and James Heather. We thank Aleks Essex and the anonymous reviewers for their input on a draft of this paper. The authors acknowledge the support of this research by the Natural Sciences and Engineering Research Council of Canada (NSERC)—the first author through a Canada Graduate Scholarship and the second through a Discovery Grant.

References

[1] B. Adida. Helios: web-based open-audit voting. *USENIX Security Symposium 2008.*

[2] B. Adida, O. de Marneffe, O. Pereira, and J.J. Quisquater. Election a university president using open-audit voting. *EVT 2009.*

[3] R.K. Aggarwal and G. Wu. Stock market manipulation—theory and evidence. *Journal of Business*, 79(4) 2003.

[4] R.J. Anderson and E. Biham. Two Practical and Provably Secure Block Ciphers: BEARS and LION. *FSE 1996.*

[5] L. Babai. Trading group theory for randomness. *STOC 1985.*

[6] B. Barak and S. Halevi. A model and architecture for pseudorandom generation and applications to `/dev/random`. *CCS 2005.*

[7] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 1973.

[8] M. Blum, P. Feldman, and S. Micali. Non-interactive zero-knowledge and its applications. *STOC 1988.*

[9] D. Boneh and M. Franklin. Identity-based encryption from the Weil pairing. *CRYPTO 2005.*

[10] R. Canetti, R. Pass and a. shelat. Cryptography from Sunspots: How to Use an Imperfect Reference String. *FOCS 2007.*

[11] D. Chaum, R. Carback, J. Clark, A. Essex, S. Popoveniuc, R. L. Rivest, P. Y. A. Ryan, E. Shen, and A. T. Sherman. Scantegrity II: end-to-end verifiability for optical scan election systems using invisible ink confirmation codes. *EVT 2008.*

[17] This is not meant to suggest that the 30 stocks used by Scantegrity are insufficient. For the number of units to be audited in this election, a seed of 16 bits expanded with a PRNG provides good statistical properties (*cf.* [12,36]). As shown, this is nearly achieved by CVX and XOM alone.

[18] Although the *security* of these ciphers has not been examined as closely as AES, neither has been examined in any detail for good *extraction* properties.

[19] Available at: `http://eprint.iacr.org/2010/361`

[12] J. Clark, A. Essex, and C. Adams. Secure and observable auditing of electronic voting systems using stock indices. *IEEE CCECE 2007*.

[13] J. Calandrino, J.A. Halderman, and E.W. Felten. In defense of pseudorandom sample selection. *EVT 2008*.

[14] J. Camenisch, S. Hohenberger, M. Kohlweiss, A. Lysyanskaya, and M. Meyerovich. How to win the clone wars: efficient periodic n-times anonymous authentication. *CCS 2006*.

[15] M. Chesney, M. Jeanblanc-Picque, and M. Yor. Brownian excursions and Parisian barrier options. *Advances in Applied Probability*, 29, 1997.

[16] A. Cordero, D. Wagner, and D. Dill. The role of dice in election audits. *WOTE 2006*.

[17] Y. Dodis, R. Gennaro, J. Hastad, H. Krawczyk, and T. Rabin. Randomness extraction and key derivation using the CBC, Cascade and HMAC modes. *CRYPTO 2004*.

[18] Y. Dodis and A. Yampolskiy. A verifiable random function with short proofs and keys. *PKC 2005*.

[19] D. Eastlake. Publicly Verifiable Nomcom Random Selection. RFC 2777, IETF, 2000. `http://www.ietf.org/rfc/rfc2777.txt`

[20] D. Eastlake. Publicly Verifiable Nominations Committee (NomCom) Random Selection. RFC 3797, IETF, 2004. `http://www.ietf.org/rfc/rfc3797.txt`

[21] A. Essex, J. Clark, R. T. Carback, and S. Popoveniuc. Punchscan in practice: an E2E election case study. *WOTE 2007*.

[22] S. Even, O. Goldreich, and A. Lempel. A randomized protocol for signing contracts. *CACM*, 28(6), 1985.

[23] R.R. Farashahi, R. Pellikaan, and A. Sidorenko. Extractors for binary elliptic curves. *Designs, Codes, and Cryptography*, 49, 2008.

[24] A. Fiat, and A. Shamir. How to prove yourself: practical solutions to identification and signature problems. *CRYPTO 1986*.

[25] S. Goldwasser and Y. Kalai. On the (in)security of the Fiat-Shamir paradigm. *FOCS 2003*.

[26] S. Goldwasser and M. Sipser. Private coins versus public coins in interactive proof systems. *STOC 1986*.

[27] J. Groth and A. Sahai. Efficient non-interactive proof systems for bilinear groups. *EUROCRYPT 2008*.

[28] A. Halderman and B. Waters. Harvesting verifiable challenges from oblivious online sources. *CCS 2007*.

[29] J.L. Hall. On improving the uniformity of randomness with Alameda County's random selection process. 2008.

[30] D. Jefferson, E. Ginnold, K. Midstokke, K. Alexander, P. Stark, and A. Lehmkuhl (State of California's Post-Election Audit Standards Working Group). Evaluation of Audit Sampling Models and Options for Strengthening Californias Manual Count. Report, 2007.

[31] A. Juels, M Jakobsson, E. Shriver, and B. Hillyer. How to Turn Loaded Dice into Fair Coins. *IEEE Transactions on Information Theory*, 46(3), 2000.

[32] R. Merton. Theory of rational option pricing. *Journal of Economics and Management Sciences*, 4(1), 1973.

[33] G. Miller. Note on the bias of information estimates. "Information Theory in Psychology II-B." Free Press, 1955.

[34] L. Paninski. Estimation of entropy and mutual information. *Neural Computation*, 15, 2003.

[35] M. Rabin. Transaction protection by beacons. *Journal of Computer and System Sciences*, 27(2), 1983.

[36] E. Rescorla. On the security of election audits with low entropy randomness. *EVT 2009*.

[37] M. Schroder. Brownian excursions and Parisian barrier options: a note. *Advances in Applied Probability*, 40(4), 2003.

[38] R.U. Seydel. "Tools for computational finance." Springer, 4th ed, 2009.

[39] B. Waters, A. Juels, J. A. Halderman, and E. W. Felten. New client puzzle outsourcing techniques for DOS resistance. *CCS 2004*.

[40] S. Wolfram. "A New Kind of Science." Wolfram Media, 1st ed, 2001.