Bayesian Post-Election Audits

Ronald L. Rivest and Emily Shen

Viterbi Professor of EECS
MIT, Cambridge, MA
{rivest,ehshen}@mit.edu

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Outline

Post-Election Audit Objectives

By examining by hand sufficiently many randomly selected paper ballots:

- Confirm to a high degree of confidence that the reported (scanner-based) outcome is correct or else that the actual (full hand-count) outcome is different.

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- Convince the losers they really lost!

Single-ballot Audits

- Sequential decision-making (Wald).
- Examine paper ballots *one at a time*, in random order.
- Determine *actual type* of each ballot (as opposed to its *reported type*).
- At each stage, decide whether to
  - Stop: Reported outcome looks **OK**.
  - Continue: more auditing needed.

  (We assume that full hand count needed to overturn reported outcome.)

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  Given what you've seen in the audit so far, what is the probability that each candidate would win if all ballots were examined?

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- Then you can stop audit if/when the reported winner has at least (say) 95% probability of winning.

Actual ballot types (by hand):

? ? ? ? ? ? ? ? ? ? . . .

Probability A wins: 50.0%

Probability B wins: 50.0%

An example

Actual ballot types (by hand):

A ? ? ? ? ? ? ? ? ?…

Probability A wins: 75.0%
Probability B wins: 25.0%

An example

Actual ballot types (by hand):

A A ? ? ? ? ? ? ? ? . . .

Probability A wins: 87.5%

Probability B wins: 12.5%

An example

Actual ballot types (by hand):

A A B ? ? ? ? ? ? ? . . .

Probability A wins: 68.8%

Probability B wins: 31.2%

An example

Actual ballot types (by hand):

A A B B ? ? ? ? ? ? . . .

Probability A wins: 50.0%

Probability B wins: 50.0%

An example

Actual ballot types (by hand):

A A B B A ? ? ? ? ? . . .

Probability A wins: 65.6%

Probability B wins: 34.4%

An example

Actual ballot types (by hand):

A A B B A A ? ? ? ? . . .

Probability A wins: 77.4%

Probability B wins: 22.6%

An example

Actual ballot types (by hand):

A A B B A A A ? ? ? . . .

Probability A wins: 85.6%

Probability B wins: 14.4%

An example

Actual ballot types (by hand):

A A B B A A A A ? ? . . .

Probability A wins: 91.0%

Probability B wins: 9.0%

An example

Actual ballot types (by hand):

A A B B A A A A A ? . . .

Probability A wins: 94.5%

Probability B wins: 5.5%

An example

Actual ballot types (by hand):

A A B B A A A A A A . . .

Probability A wins: 96.7%

Probability B wins: 3.3%

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$\rightarrow$ Stop auditing! $\leftarrow$

Making the magic box (ballot-polling)

- Suppose you are auditing an election between candidates A and B, with 5 ballots.

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- You draw a random sample (without replacement) of two ballots.
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- *Q: What is the probability that A won?*

Answer (Bayesian)

- To make Q well-posed, need a model (a *prior*) for the likelihood of different outcomes.

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- A *noninformative prior* gives each outcome (A:B tally) equal probability:

tally	5:0	4:1	3:2	2:3	1:4	0:5
Prob	1/6	1/6	1/6	1/6	1/6	1/6

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- If your error limit is 5%, stop auditing!

95%? (Bayes Rule)

posterior probability proportional to:

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tally	5:0	4:1	3:2	2:3	1:4	0:5
prior	1/6	1/6	1/6	1/6	1/6	1/6
likelihood(AA)	$\frac{5}{5} \cdot \frac{4}{4}$	$\frac{4}{5} \cdot \frac{3}{4}$	$\frac{3}{5} \cdot \frac{2}{4}$	$\frac{2}{5} \cdot \frac{1}{4}$	0	0

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product	$\frac{10}{60}$	$\frac{6}{60}$	$\frac{3}{60}$	$\frac{1}{60}$	0	0

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tally	5:0	4:1	3:2	2:3	1:4	0:5
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likelihood(AA)	$\frac{5}{5} \cdot \frac{4}{4}$	$\frac{4}{5} \cdot \frac{3}{4}$	$\frac{3}{5} \cdot \frac{2}{4}$	$\frac{2}{5} \cdot \frac{1}{4}$	0	0
product	$\frac{10}{60}$	$\frac{6}{60}$	$\frac{3}{60}$	$\frac{1}{60}$	0	0
posterior	$\frac{10}{20}$	$\frac{6}{20}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0

95%? (Bayes Rule)

posterior probability proportional to:

prior times likelihood of sample given prior

tally	5:0	4:1	3:2	2:3	1:4	0:5
prior	1/6	1/6	1/6	1/6	1/6	1/6
likelihood(AA)	$\frac{5}{5} \cdot \frac{4}{4}$	$\frac{4}{5} \cdot \frac{3}{4}$	$\frac{3}{5} \cdot \frac{2}{4}$	$\frac{2}{5} \cdot \frac{1}{4}$	0	0
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posterior	$\frac{10}{20}$	$\frac{6}{20}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0
	A wins 95%			B wins 5%		

Efficient sampling of posterior using Polya's Urn

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6. Tally the n balls in the urn.

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   Urn = { $\underbrace{\text{A A}}_{\textit{sample}}$ A B A } $\Longrightarrow$ tally 4:1
3. Add s balls for sample (e.g. A A)
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   Tally has desired distribution! (e.g. $\frac{10}{20}, \frac{6}{20}, \frac{3}{20}, \frac{1}{20}$)

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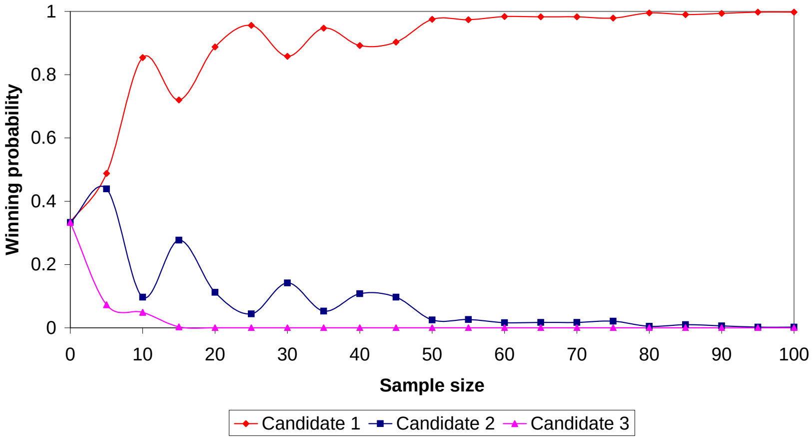
Can sample faster using gamma variates (see paper).

Winning probabilities vs. sample size in a Bayes audit



Arbitrary voting system

We note that a Bayes audit works for an *arbitrary voting system* as long as the number of ballot types is not too large; all you need is a way to compute the winner of a profile of ballots, and a way of sampling ballots. We have tested it on

- plurality
- IRV
- Borda
- Schulze

with good results.

Bayesian comparison audits

- Same idea, but have *one urn for each reported type*.
- Much more efficient!! (But needs way of matching paper ballots with their reported types.)

2011 Monterey Peninsula Water Mgt District Director

- Ballot-polling.
- Two candidates (plus write-ins).
- 2011 votes cast: 1353 for Lewis, 742 for Mancini (reported).
- Stark's ballot-polling audit with 10% risk limit examined:

  89 ballots.

- A Bayes ballot-polling audit with $\epsilon = 0.10$ examines:

  23 ballots on average

  11 ballots (median)

2011 Stanislaus Oakdale Measure O

- Comparison audit.
- Yes/No proposition.
- 3152 votes cast: 1728 Yes, 1392 No, 32 undervotes (reported).
- Stark's comparison audit with 10% risk limit examined:

  49 ballots.
- A Bayes ballot-polling audit with $\epsilon = 0.10$ examines:

  92 ballots (average)

  39 ballots (median).

Discussion

- We conjecture that a Bayes audit is in fact "risk-limiting" (perhaps given some suitable assumptions or constant factors in parameterization). But this is just a conjecture.
- The Bayes audit admits the use of other priors, such as those a very partisan observer might have.
- The Bayes audit admits the use of *multiple* priors; only stopping when all auditors (with different priors) agree to do so.

Summary – Bayes Audit Advantages

- High efficiency (few ballots get audited).
- Small/controllable miscertification rates observed.
- Simple in structure / easy to implement.
- Handles ballot-polling audits, comparison audits, and many different voting systems.
- No MOV computation required to start.
- Admits flexible (multiple) choice(s) of prior.
- Can be stopped early with meaningful results.

Summary – Bayes Audit Disadvantages

- Only works (so far) for single-ballot audits.
- Unclear relationship to risk-limiting audits.
- Results depend on choice(s) for prior.
- Need program to compute winning probabilities.

The End

For more info and code, contact authors or see:

`http://people.csail.mit.edu/rivest/bayes/`